Magnetic train

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29.6.2016.

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Contents

1 LIST OF SYMBOLS

SYMBOL	MEASUREMENT UNIT	MEANING
B_z	T	coil's magnetic field
B_i	T	magnetic field of one coil turn i
m_i	Nm/T	magnetic dipole moment of one coil turn i
l or d	cm	lenght of the part of the coil carrying current, ie. lenght of train
R	cm	coil radius
L	H	inductance of the part of the coil carrying current
R_Ω	Ω	ohmic resistance of the part of the coil carrying current
Φ_i	Wb	magnetic flux through one coil turn i
F_z	N	force of coil on magnet
B_{mi}	T	magnetic field of one magnet
m_m	Nm/T	magnet's magnetic dipole moment
F_m	N	force of magnet on the coil
F_p	N	magnet's force of holding on the battery
F_{od}	N	repulsion force between magnets
μ_0	H/m	magnetic permeability
I_{ot}	A	current induced due to resistance of coil towards the passage of magnets
I	A	short circuit current of battery
r_i	cm	distance between centre of magnet and coil turn i
F_{ot}	N	coil's force of resistance towards the passage of magnets
F_{tr}	N	friction force
μ		coefficient of dynamic friction
s	cm	distance travelled
v	cm/s	velocity
v_k	cm/s	terminal velocity
t	s	time

2 INTRODUCTION

Button magnets are placed on both sides of a cylindrical battery. Magnets together with the battery make up the train which starts to move if it's placed in a coil made from uninsulated wire such that the magnets contact the wire.
During literature review, I have not found any work concerning itself with this or similar phenomena.
My purpose in this work is to explain the phenomenon and investigate the influence of relevant parameters on the train's terminal velocity.

3 THEORETICAL MODEL

By putting the train in the coil, the electric circuit gets closed, as illustrated on fig. 1, and currents starts to flow through the part of the coil where the magnet is situated. Since magnetic current is induced around current-carrying conductor, the part of the coil where the train is situated becomes an electromagnet. Interaction between this electromagnet and the two button magnets causes motion. Since current flows from the positive pole of the battery towards the negative one only through the part of the coil where the train is situated, the magnetic field of the coil induced by this current moves together with the train. Because the ends of the whole experimental coil are not connected in any manner, the parts of the coil in which the train is not situated at the moment of motion can be ignored since no significant current which might affect the phenomenon in a measurable manner can be induced.

At the incipient moment, we can ignore the magnetic field component induced by coil's resistance towards the passage of magnets. It's important to notice that the magnets are situated at the end of the electromagnet where the magnetic filed is inhomogeneous. If they were situated in the centre of the electromagnet, they wouldn't feel any force, as we can see from the expression describing the coil's force on a magnet $\overrightarrow{F_z} = \nabla(\overrightarrow{m_m} \cdot \overrightarrow{B_z})$ [1]. In such a configuration, there could only be a torque present on the magnets if the magnetic dipole moments of the coil and the magnets weren't collinear, but no translational motion would occur. So the influence of the coil on the magnets wouldn't cancel out, it is necessary to orient magnets oppositely. The configuration should be either N-S...S-N or S-N...N-S. When oriented like that, the coil's magnetic field will push one, and pull the other magnet. Fig. 2 shows the magnets, the part of the coil conducting current and the forces arising from their interaction.

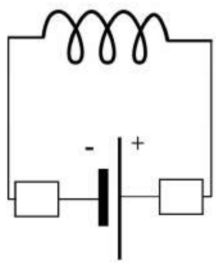


Figure 1: equivalent circuit of the train and the part of the coil carrying current

Even though the magnets have the same poles oriented towards each other, the train is a static system because the magnet's force of holding on the battery is greater than the repulsion force between the magnets: $\overrightarrow{F_p} > \overrightarrow{F_{od}}$. Interaction between the magnets does not influence motion, so it will not be considered further in this work.

Because of the 3rd Newton's law, the force of magnets on the coil is equal in magnitude and opposite in direction as the force of the coil on the magnets $\overrightarrow{F_m} = -\overrightarrow{F_z}$. $\overrightarrow{F_m}$ is Amper's force and is given by $\overrightarrow{F_m} = \sum_{i=1}^{N} I\,\overrightarrow{l} \times \overrightarrow{B_{mi}}$, where i represents a single coil's turn and l represents a turn's length of wire $l = 2R\pi$. Direction of this force is determined with the right hand rule.

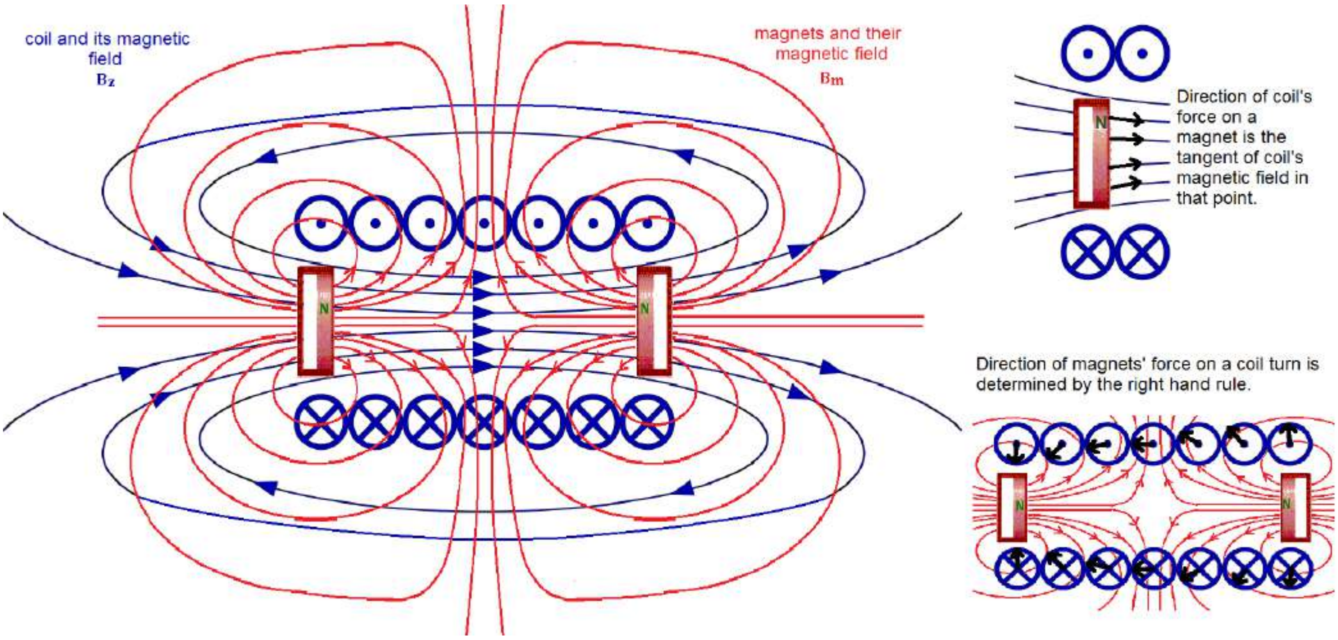


Figure 2: sketch of coil's magnetic field (blue) [3], magnets' magnetic field (red) and forces they cause (black arrows)

If we observe any moment after the incipient moment of motion, we will notice an additional component of the magnetic field. Passage of the magnets causes change of flux through the coil. So because of Lenz's rule, current is induced in the coil. This current is negligible outside of the electromagnet because the coil as a whole is an open circuit. Because of the law of conservation of energy, this current will necessarily flow such that it lessens the change of the magnetic flux, ie. it will flow from the negative to the positive pole of the battery and decrease the total strength of current flowing through the coil. The faster the train is moving, the smaller the current flowing through the coil and thus the weaker the electromagnet and the force causing movement. We can conclude that, since the faster the train is moving, the greater resistance towards its movement it will encounter, the train must reach some terminal velocity.

The direction of the train's movement is influenced by the orientation of magnets, the battery and the coil turns. Changing the orientation of any of these components will change the relative orientation of the magnets and the electromagnet, and this will change the direction of movement of the train.
Fig. 3 shows a diagram of all the forces in the system.

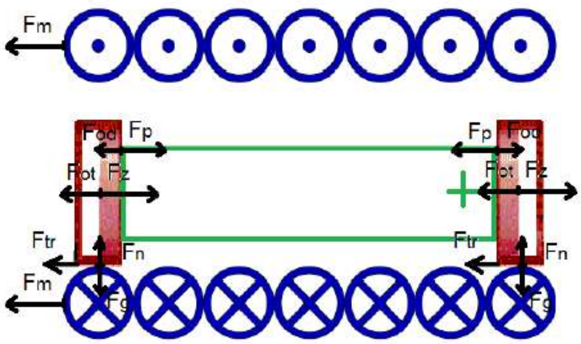


Figure 3: force diagram

3.1 Theoretic model

The main approximations of this model is that the lines of symmetry of the magnets and the coil coincide and that the currents induced outside of the electromagnet are negligible.

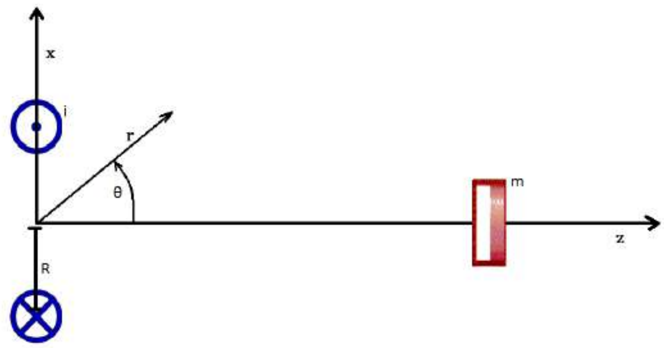


Figure 4: coordinate system

The coordinate system is chosen so that the magnetic dipole moment of a coil turn carrying current m_i is oriented towards z. The system is placed so that the magnetic dipole moment of the coil turn and that of the magnet m_m lay on the same line ($\theta = 0$). Looking at one coil turn carrying current i, we can define its magnetic field as:

$$\overrightarrow{B_i}(z) = \frac{\mu_0 I}{2}\left(\frac{R^2}{(R^2 + {r_i}^2)^{3/2}}\right)\hat{z}$$

[1], where r is the distance between the centre of the magnet and the coil turn, I the short circuit current of the battery and R coil radius.

Now let's look at the coil's force on the magnets. Because of the radial symmetry of the system, force components in the x and y direction will cancel out. We are interested only in the force component in the z direction, so the gradient in the expression degenerates into a partial derivation of the product $\overrightarrow{B_i} \cdot \overrightarrow{m_m}$ over z:

$$\overrightarrow{F_i} = \nabla(\overrightarrow{B_i} \cdot \overrightarrow{m_m}) = \frac{\partial}{\partial z}\left(\frac{\mu_0 I}{2} \frac{R^2 \overrightarrow{m_m} \cdot \hat{z}}{(R^2 + {r_i}^2)^{3/2}}\right)\hat{z} = \frac{-3\mu_0 I R^2 m_m}{2} \frac{r_i}{(R^2 + {r_i}^2)^{5/2}}\hat{z}.$$

We then sum the influence of n coil turns carrying current on every magnet:

$$\overrightarrow{F_z} = 2 \cdot \sum_{i=1}^{n} \overrightarrow{F_i} = 2 \cdot \sum_{i=1}^{n} \frac{-3\mu_0 I R^2 m_m}{2} \frac{r_i}{(R^2 + {r_i}^2)^{5/2}}\hat{z} = -3\mu_0 I R^2 m_m \hat{z} \cdot \sum_{i=1}^{n} \frac{r_i}{(R^2 + {r_i}^2)^{5/2}}.$$

This force is responsible for the movement of the train.

Also important are the friction force:

$$\overrightarrow{F_{tr}} = mg\mu,$$

where m is the mass of the train and μ the coefficient of dynamic friction; and the coil's force of resistance towards the passage of the magnets, ie. the change of the magnetic flux in the coil: $\overrightarrow{F_{ot}}$. According to Faradey's law, the coil resists the change of magnetic flux by inducing current I_{ot} such that its magnetic field lessens the change:

$$\epsilon_{ot} = -L\frac{d\Phi}{dt} = -I_{ot} R_\Omega$$

[2], where R_Ω is ohmic resistance of the electromagnet. By similar procedure as for the force causing the movement, the expression for the main force limiting it can be obtained:

$$\overrightarrow{F_{ot}} = 2 \cdot \sum_{i=1}^{n} \frac{-3\mu_0 I_{ot} R^2 m_m}{2} \frac{r_i}{(R^2 + {r_i}^2)^{5/2}}\hat{z} = -3\mu_0 R^2 m_m \hat{z} \cdot \sum_{i=1}^{n} \frac{I_{ot} r_i}{(R^2 + {r_i}^2)^{5/2}}$$

$$= -3\mu_0 R^2 m_m \hat{z} \cdot \sum_{i=1}^{n} \frac{-L}{R_\Omega} \frac{d\Phi_i}{dt} \frac{r_i}{(R^2 + {r_i}^2)^{5/2}}$$

$$= 3\mu_0 R^2 m_m \frac{L}{R_\Omega}\hat{z} \cdot \sum_{i=1}^{n} \frac{d\Phi_i}{dt} \frac{r_i}{(R^2 + {r_i}^2)^{5/2}}.$$

It is determined by experiment that the train moves with constant velocity after at most 0.15s after the start of the movement. In the beginning, $\overrightarrow{F_z}$ overcomes $\overrightarrow{F_{tr}}$ and the resultant force causes the acceleration of the train; but almost immediately $\overrightarrow{F_{ot}}$ arises and the faster the train is moving, the bigger it is. It acts like breaks on the train. With constant work of the battery, a balance is achieved where the resultant force on the train is zero $\overrightarrow{F_z} = \overrightarrow{F_{tr}} + \overrightarrow{F_{ot}}$.
Velocity may be extracted by simple substitution from the sum of flux change in every coil turn:

$$\sum_{i=1}^{n} \frac{d\Phi_i}{dt} = \sum_{i=1}^{n} \frac{d\Phi_i}{dt}\frac{dz}{dz} = \sum_{i=1}^{n} v\frac{d\Phi_i}{dz} = v\sum_{i=1}^{n} \frac{d\Phi_i}{dz}.$$

Substituted into the force equation:

$$-3\mu_0 R^2 m_m \hat{z} \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}} (I - \frac{L}{R_\Omega} v \sum_{i=1}^{n} \frac{d\Phi_i}{dz}) + mg\mu\hat{z} = -ma\hat{z}$$

we obtain a first order differential equation:

$$m\dot{v} + v \cdot 3\mu_0 R^2 m_m \frac{L}{R_\Omega} \sum_{i=1}^{n} \frac{d\Phi_i}{dz} \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}} - 3\mu_0 R^2 m_m I \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}} + mg\mu = 0.$$

The solution of this equation is:

$$v(t) = Ke^{-3\mu_0 R^2 m_m \frac{L}{mR_\Omega} \sum_{i=1}^{n} \frac{d\Phi_i}{dz} \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}}} + \frac{3\mu_0 R^2 m_m I \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}} - mg\mu}{3\mu_0 R^2 m_m \frac{L}{R_\Omega} \sum_{i=1}^{n} \frac{d\Phi_i}{dz} \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}}}$$

And since the boundary condition is that the train is stationary at the incipient moment:

$$K = -\frac{3\mu_0 R^2 m_m I \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}} - mg\mu}{3\mu_0 R^2 m_m \frac{L}{R_\Omega} \sum_{i=1}^{n} \frac{d\Phi_i}{dz} \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}}}.$$

If we let time go into the limit of positive infinity, we will obtain terminal velocity as:

$$v_k = \frac{1}{\frac{L}{R_\Omega} \sum_{i=1}^{n} \frac{d\Phi_i}{dz}} (I - \frac{mg\mu}{3\mu_0 m_m R^2 \cdot \sum_{i=1}^{n} \frac{z_i}{(R^2 + {z_i}^2)^{5/2}}}).$$

4 EXPERIMENTAL SET UP AND METHODS

4.1 Experimental set up

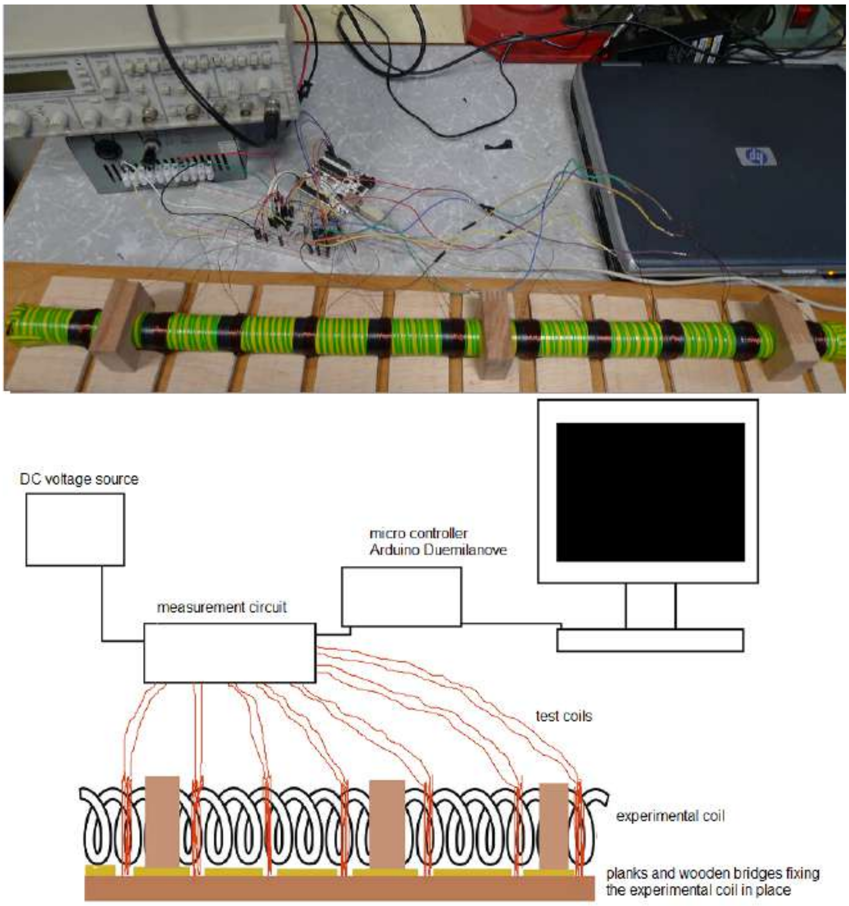


Figure 5: picture and sketch of experimental set up

Figure 6: coils used

My set up primarily consisted of 3 coils, shown on fig.6, wound such that the distance between two turns was 0.8mm. They were covered with two layers of insulating tape so that they wouldn't unwind, deform from the force of the magnets on the coil or short circuit themselves. Thickness of wire was 1.2mm and turn density $510m^{-1}$. The three coils only differed in radius: the main one was 2.5cm, and the other two were 2.85cm and 3.6cm. The greatest problem was ensuring that the coils don't short circuit. Methods of inserting paper or insulating tape between coil turns have proven unsuccessful because it was impossibly hard to make sure they don't come in contact with the train. Method of wounding alternating uninsulated and lacquered wire has also proven unsuccessful because the train would lose contact with the uninsulated wire.
While conducting experiments, the coil was fixed to a plank to ensure minimal deformation. Smaller test coils, 50 turns each, were wound over coils to detect the passage of the magnets. Ten were wound over the main coil, and 5 over the other two.

Every time a magnet would pass through a test coil, it would induce a small momentary voltage. The polarity of this voltage would switch when the magnet was leaving the coil as compared to when it was entering it (passage signal shown on fig.9). Since the micro-controller used could only

accept input 0-5V, operational amplifiers were used to offset the signal. A multiplexor was used to increase the number of input channels on the micro-controller. Every 3 ms, micro-controller would read the value of voltage on a different test coil and print out this reading on the computer. Code used is in the addendum. Measurement circuit is shown on fig.7.
Common AA alkaline 1.5V batteries and neodymium NdFeB magnets coated with Ni, 2.5mm thick and 2cm in radius with 0.2T surface magnetic field, were used for the train. Measurement results were processed in SigmaPlot.

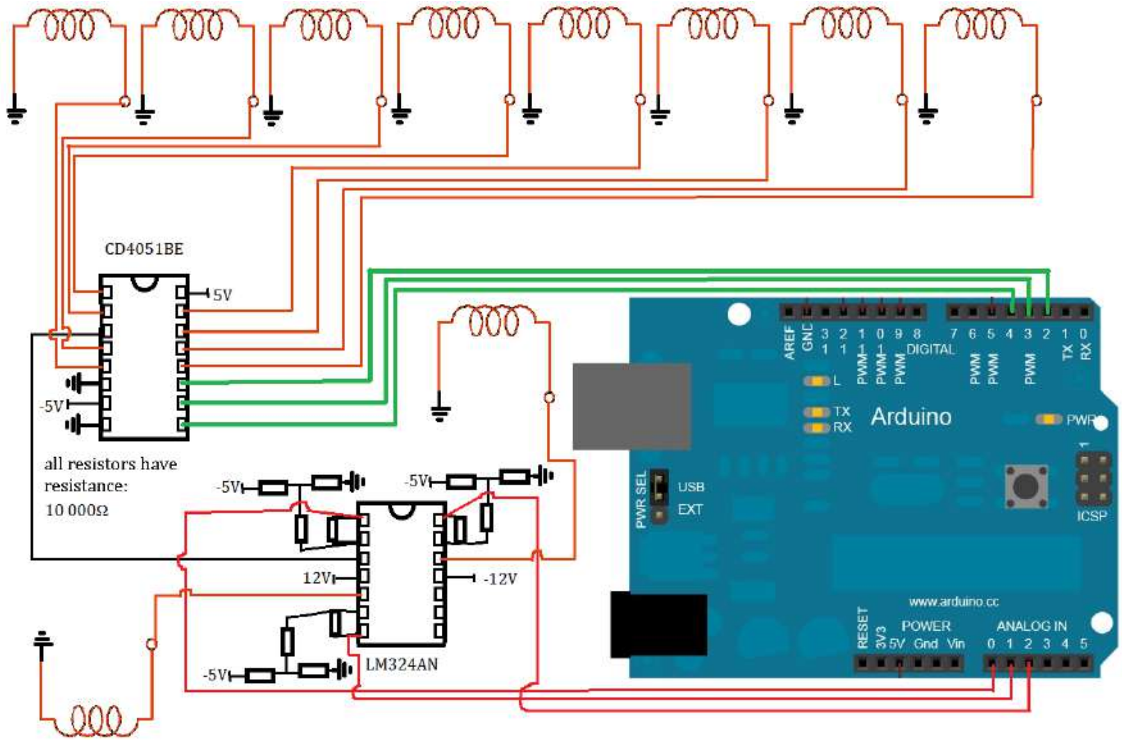


Figure 7: measurement circuit [4]

Since the train is already moving with constant velocity by the time it reaches the first coil, motion at the incipient was investigated with a separate set up (shown on fig.8). High speed camera filming 120fps was placed above the entrance to the coil and a marked ribbon was fixed to the train. Recordings were processed in Tracker.

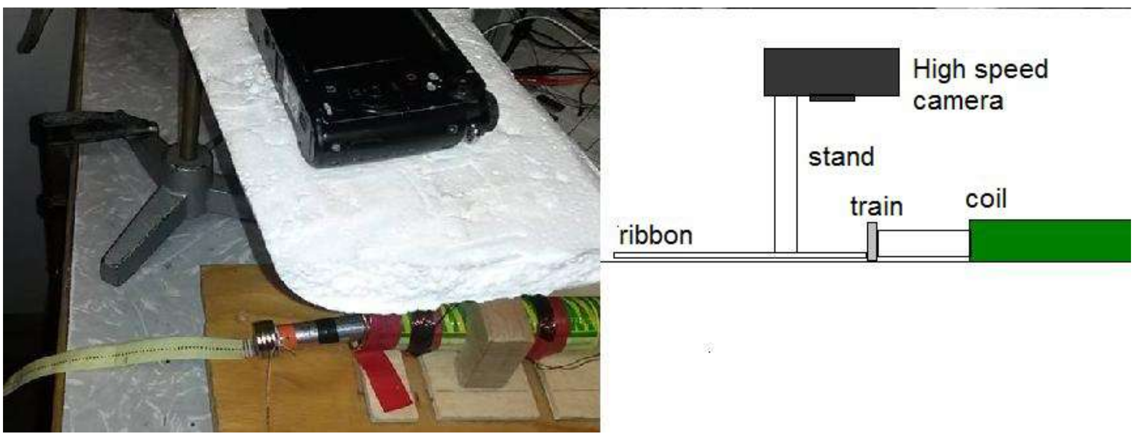


Figure 8: experimental set up for the analysis on the onset of movement

4.2 Parameters

Terminal velocity's dependence on parameters determining the coil's force on magnets which primarily causes motion and accelerates the train was experimentally investigated. These parameters determine the strength of magnets (surface magnetic field), strength of electromagnet (short circuit current of the battery) and geometry of electromagnet's magnetic field (length, coil radius). Parameters affecting motion, but not directly connected with the source of phenomenon were kept constant (mass, coefficient of friction).

4.3 Methods

Magnets' strength was altered by stacking more button magnets on each side. When measuring dependence on surface magnetic field, to keep mass and length of the train constant, copper cylinders with mass equivalent to mass of two magnets (11.8g) and 5mm high were inserted between the battery and the magnets. To ensure that they are always in contact with the same number of coil turns, 2 magnets on the edges were covered with 3 layers of aluminium foil, while the rest had outside edges covered with insulating tape. The case with an unequal number of magnets stacked on each side was also tested.
Short circuit current of the battery was checked with an analogue ammeter during every measurement to ensure that this parameter was kept constant. Short circuit current of the battery was simply varied by using batteries at different stages of life cycle.
When measuring dependence on train length, small plexiglas cuboids of negligible mass (0.7g each) thick 5mm were inserted between the battery and the magnets with a strip of aluminium foil to ensure contact.
Coil radius was varied by winding up additional coils.
Dynamic coefficient of friction between the magnets and the coil was determined by pulling a non-magnetised Ni coated ball with a spring-scale with precision of 0.02N through the coil. It was determined to be equal to $\mu = 0.22$.
5 measurements were made for every point on the graphs. Error bars represent the maximum difference from the average value. A new battery was used for every set measurements to ensure comparability of the results.

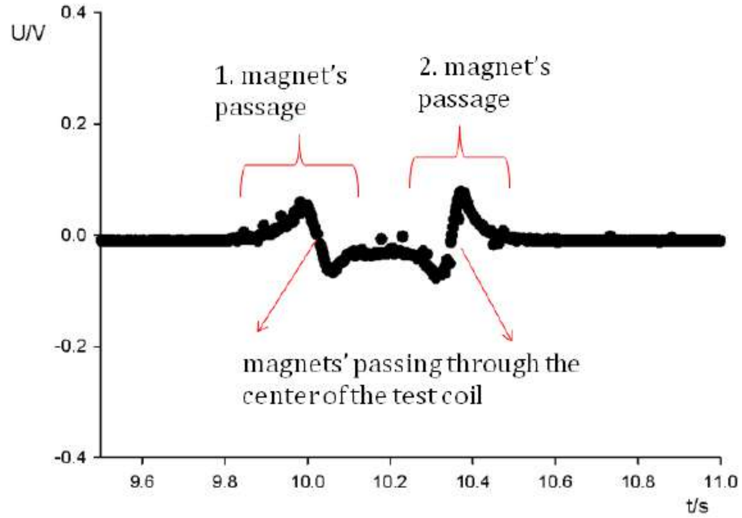
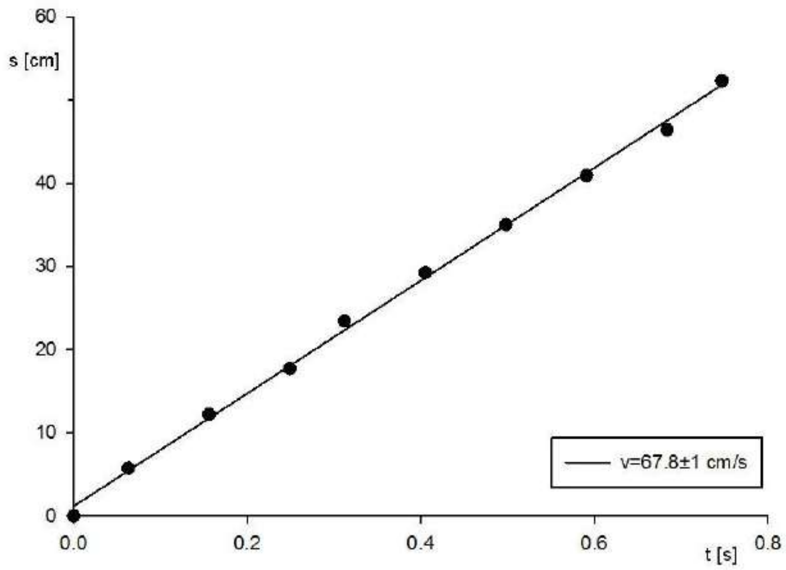


Figure 9: Passage of the train captured with an oscilloscope measuring voltage on one test coil. The moment of the first peak is taken as the moment of passage of the train through the test coil.



surface magnetic field: 0.8T, short circuit current: 3.6A, length: 7cm, coil radius: 2.5cm

Figure 10: measurement example: s-t graph

5 MEASUREMENT RESULTS

5.1 Motion at the incipient

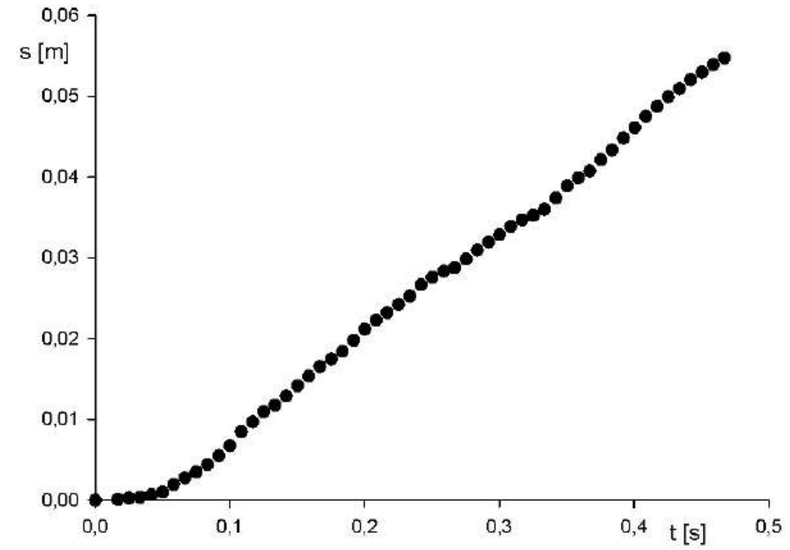


Figure 11: s-t graph of train's motion captured with a high speed camera

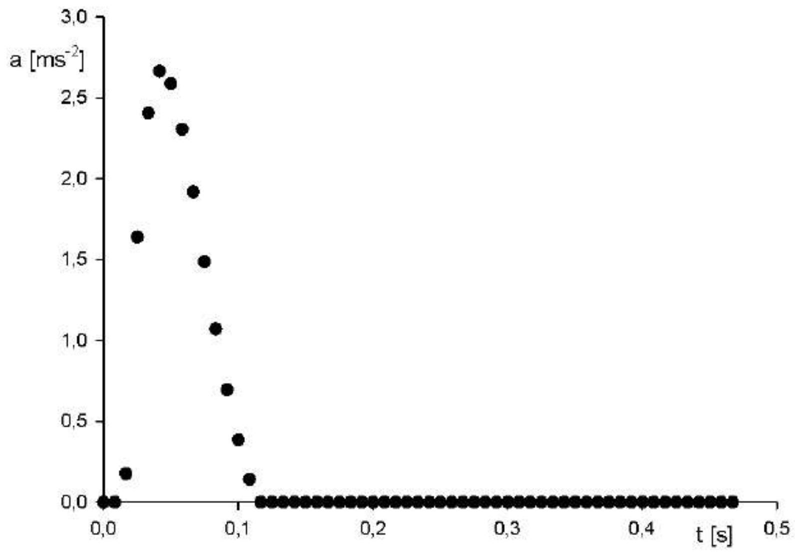


Figure 12: a-t graph of train's motion obtained by numerical derivation of the s-t graph

S-t graph in fig.11 shows how the train accelerates at the very beginning and reaches terminal velocity in a short amount of time. Deviations from a straight line after 0.15s are present only because of the imperfectness of the measurement method. A-t graph in fig.12 shows how the train reaches maximum acceleration almost instantly (as soon as the whole train is inside the coil) and then how it falls to zero. This tells us that the resultant acceleration, ie. force, is zero during all motion except at the very beginning.

5.2 Dependence on short circuit current

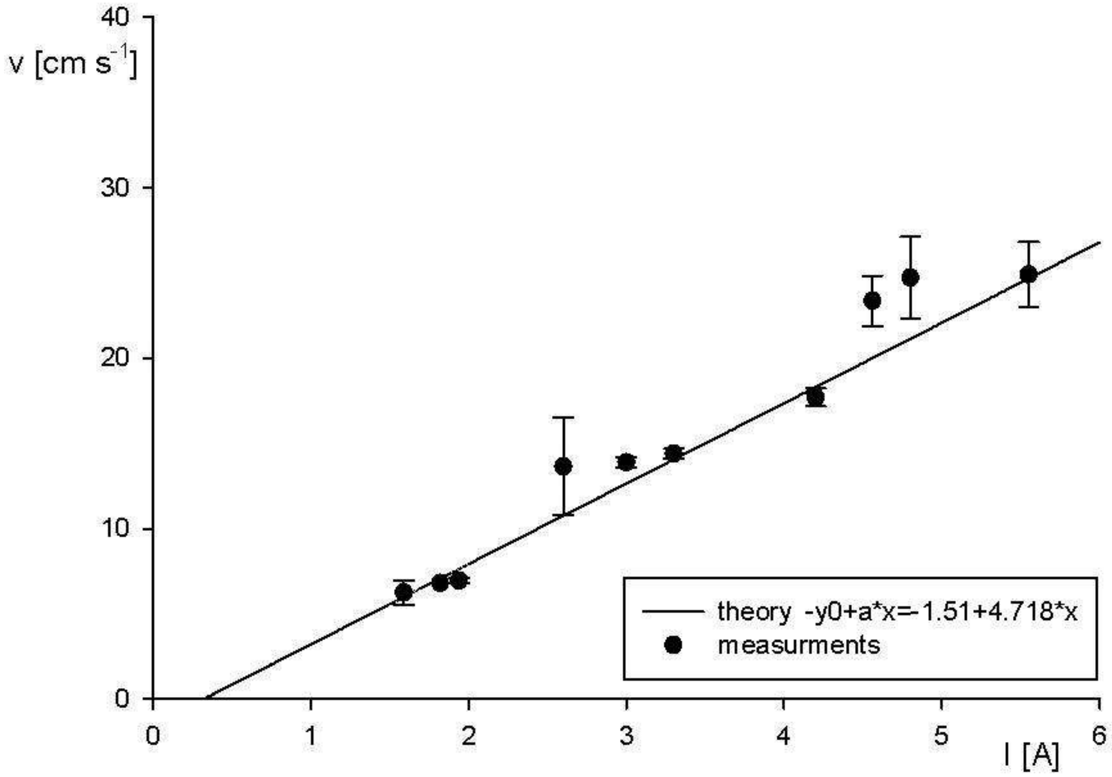


Figure 13: graph of terminal velocity's dependence on battery's short circuit current

Fig.13 clearly shows positive linear dependence of terminal velocity on battery's short circuit current as predicted by the theory. Because the train has to first overcome friction to start moving, the minimum current required for the start of movement is non-zero.
There is greater error around the value of 5A because that is the current new batteries give and in that stage of life cycle current strength falls the fastest with use. The rest of the error is caused by the imperfectness of the ammeter.

5.3 Dependence on surface magnetic field of the magnets

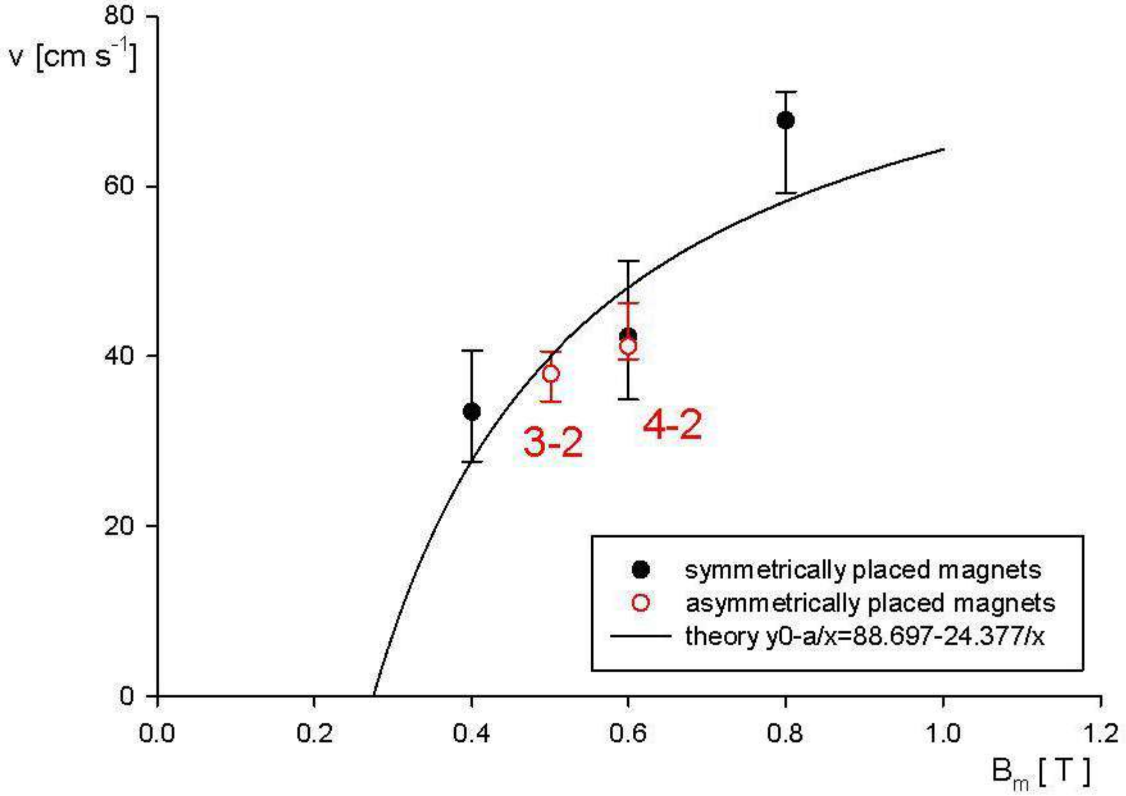


Figure 14: graph of terminal velocity's dependence on magnet's surface magnetic field strength

For symmetrically placed magnets, the value B_m shown represents the surface magnetic field of magnets on one side of the battery, while for the asymmetrically placed it represents the arithmetic average of the surface magnetic field of the magnets on both sides of the battery. Red numbers represent the number of button magnets stacked on each side of the battery in asymmetrical cases. Which side has more button magnets does not influence the movement because of the principle of superposition.
On fig.14 we can see positive dependence of the terminal velocity on magnet's surface magnetic field. We can also see that the more magnets there are, the smaller effect new addition will have on the terminal velocity. This is because these new magnets will be further away from the electromagnet's edge and will so experience a smaller force.
Magnet's surface magnetic field is a function of m_m such that $B_m \sim m_m$. From the expression for the train's terminal velocity, we can see that the terminal velocity's dependence on magnets' surface magnetic field is of the form $v_k = C_1 - \frac{C_2}{m_m} \sim y_0 - \frac{a}{B_m}$, where C_1, C_2, y_0 and a are constants. On the graph, we can see agreement between theory and experimental results. The main source of error is imperfectness of battery.
The phenomenon is not caused by the shape of the magnets, the train behaves the same when spherical magnets are used instead of button ones, though button magnets are more easily stacked to amplify the effect.

5.4 Dependence on electromagnet length

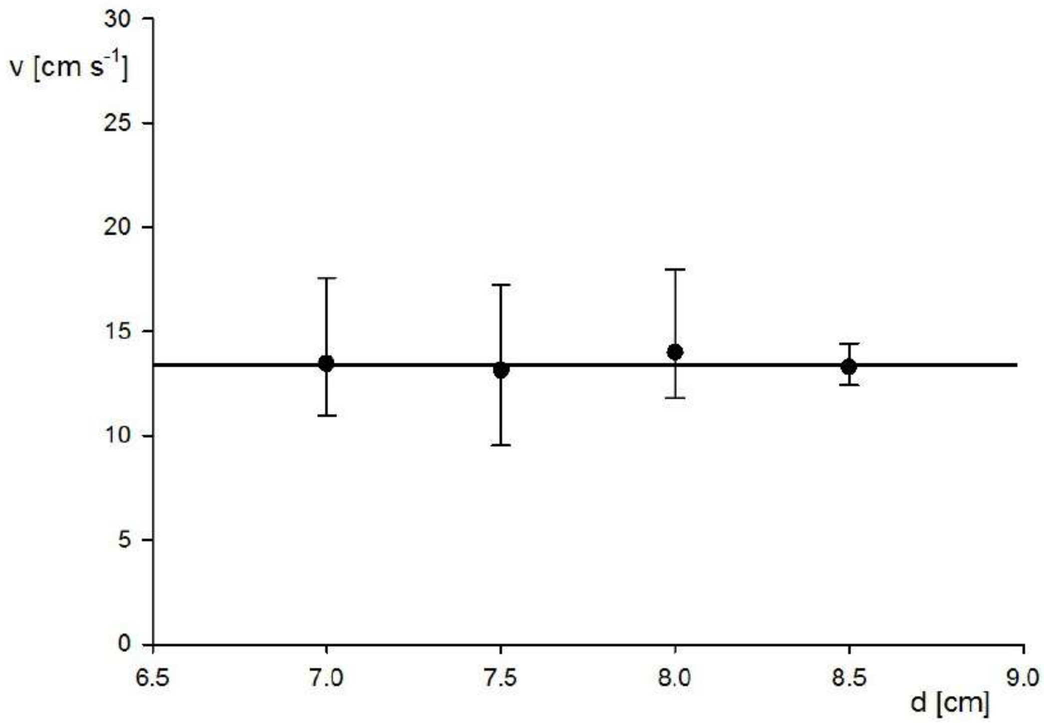


Figure 15: graph of terminal velocity's dependence on electromagnet length

Error is present because of the imperfectness of the battery.
d is the maximum distance between coil turn and magnet $z_n = d$
Looking at the force equation, we can see that the resultant force is equal to the sum of interaction of every current carrying coil turn and every magnet. Because of the factor $\sum_{i=1}^{n} \frac{z_i}{(R^2+{z_i}^2)^{5/2}}$, the effect of every coil turn is significantly smaller that the effect of its neighbouring coil turn closer to a magnet. This fact is so significant that there is no measurable effect for the changes in electromagnet length of the orders of magnitude I have tested. The phenomenon is isolated to the edges of the electromagnet.

5.5 Dependence on coil radius

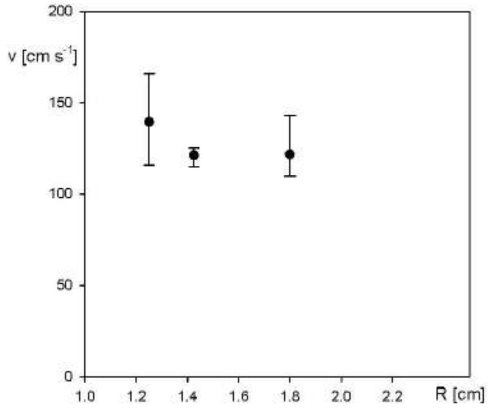


Figure 16: graph of terminal velocity's dependence on coil radius

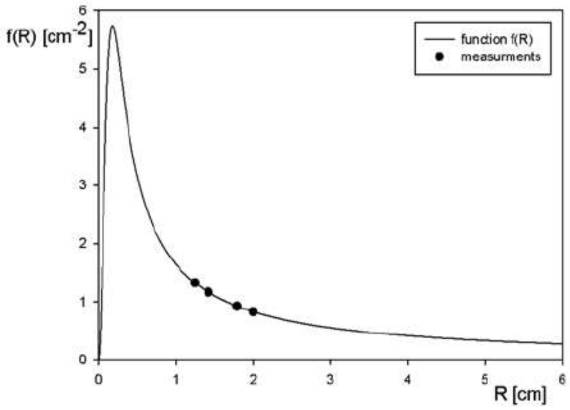


Figure 17: measurement parameters R as arguments of function $f(R) = R^2 \sum_{i=1}^{n} \frac{z_i}{(R^2+{z_i}^2)^{5/2}}$

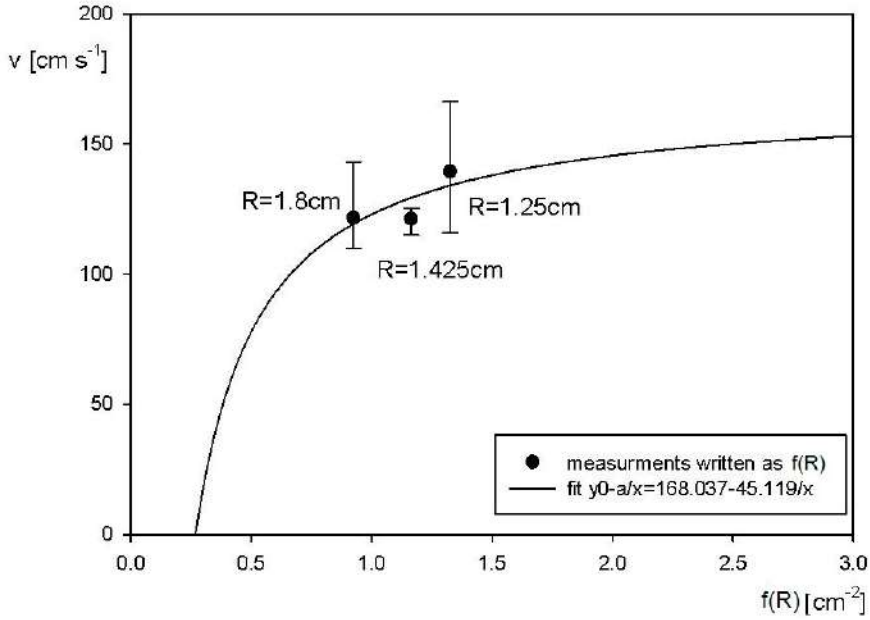


Figure 18: graph of terminal velocity's dependence on $f(R)$

The dependence of terminal velocity on coil radius is complicated. To make measurement results clearer in comparison to theory, coil radiuses from fig.16 were replaced with the function $f(R) = R^2 \sum_{i=1}^{n} \frac{z_i}{(R^2+{z_i}^2)^{5/2}}$ of them in fig.18. That function is plotted in fig.17 and experimental parameters of R are also shown on the same figure. From the expression for the train's terminal velocity, we can see that the terminal velocity's dependence on coil radius is of the form $v_k = C_1 - \frac{C_2}{f(R)}$, where C_1 and C_2 are constants. On the graph, we can see agreement between theory and experimental results. The main source of error is imperfectness of the coils.

6 CONCLUSION

A theoretical model with qualitative predictive power was developed to explain the motion of the train as an interaction between magnets and an electromagnet formed by the closing of the electric circuit.
By experimental examination, train's terminal velocity's was found to be positively dependant on battery's short circuit current and magnets' surface magnetic field, no dependence was found on electromagnet length and negative dependence was found on coil radius.
Errors in measurements were caused by the imperfectness of the set up.

7 ACKNOWLEDGES

I would like to thank Istraživački centar mladih for lending of equipment which I couldn't be able to access otherwise and support.

8 REFERENCES

[1] David J. Griffiths: *Introduction to electrodynamics, 3rd ed.;pages 218., 258.* (2001)

[2] Edward M. Purcell: *Electricity and magnetism, 2nd ed.; page 281*

[3] *coil's magnetic field diagram was downloaded and modified from http://en.academic.ru/pictures/enwiki/83/Solenoid_Rotated.svg 29.6.2016.*

[4] *micro-controller picture used in the schematics was downloaded and modified from http://duino4projects.com/wp-content/uploads/2013/07/schematic.jpg 29.6.2016.*

9 ADDENDUM: CODE FOR MICRO-CONTROLLER

```
#include "TimerOne.h"
volatile int i=0;

void setup() {
    Serial.begin(115200);
    pinMode(2, OUTPUT);
    pinMode(3, OUTPUT);
    pinMode(4, OUTPUT);
    Timer1.initialize(3000);
    Timer1.attachInterrupt(citanje);
}

void salji(int i){
    switch (i) {
        case 0:
            digitalWrite(2, LOW);
            digitalWrite(3, LOW);
            digitalWrite(4, LOW);
            break;
        case 1:
            digitalWrite(2, HIGH);
            digitalWrite(3, LOW);
            digitalWrite(4, LOW);
            break;
        case 2:
            digitalWrite(2, LOW);
            digitalWrite(3, HIGH);
            digitalWrite(4, LOW);
            break;
        case 3:
            digitalWrite(2, HIGH);
            digitalWrite(3, HIGH);
            digitalWrite(4, LOW);
            break;
        case 4:
            digitalWrite(2, LOW);
            digitalWrite(3, LOW);
            digitalWrite(4, HIGH);
            break;
        case 5:
            digitalWrite(2, HIGH);
            digitalWrite(3, LOW);
            digitalWrite(4, HIGH);
            break;
        case 6:
            digitalWrite(2, LOW);
            digitalWrite(3, HIGH);
            digitalWrite(4, HIGH);
            break;
        case 7:
            digitalWrite(2, HIGH);
            digitalWrite(3, HIGH);
            digitalWrite(4, HIGH);
            break;
```

```
    }
}

void citanje(){
    salji(i);
    switch (i) {
        case 0:
            Serial.print(analogRead(0));
            i+=1;
            break;
        case 1:
            Serial.print(analogRead(0));
            i+=1;
            break;
        case 2:
            Serial.print(analogRead(0));
            i+=1;
            break;
        case 3:
            Serial.print(analogRead(0));
            i+=1;
            break;
        case 4:
            Serial.print(analogRead(0));
            i+=1;
            break;
        case 5:
            Serial.print(analogRead(0));
            i+=1;
            break;
        case 6:
            Serial.print(analogRead(0));
            i+=1;
            break;
        case 7:
            Serial.print(analogRead(0));
            i+=1;
            break;
        case 8:
            Serial.print(analogRead(1));
            i+=1;
            break;
        case 9:
            Serial.print(analogRead(2));
            i+=1;
            break;
    }
    Serial.print("\t");
    if(i==10){
        i=0;
        Serial .print("\n");
    }
}
void loop() {
}
```